Positive Feedback May Sometimes Promote Stability

Terrence Bone
University of Washington
Seattle, Washington

Submitted by Richard A. Brualdi

ABSTRACT

Some sufficient conditions are given for a matrix to be potentially stable. These are used to furnish some counter-intuitive examples of stability behavior.

INTRODUCTION

It is widely believed that in a system of interacting elements negative feedback enhances the stability of the system, while positive feedback disturbs it. I cite two eminent examples: R. H. May, in discussing population fluctuations in an ecosystem, refers to the "destabilizing positive feedback in its intraspecific interactions" [5]. James Quirk, writing in the context of price fluctuations near a competitive equilibrium, says that if a system with some positive self-loops is stable for some magnitude of the relevant coefficients, then the related system with the positive self-loops removed is also stable for some magnitude of the coefficients [6].

This paper presents examples which counter or qualify these assertions and show that in some cases positive feedback is essential to stability, while negative feedback destroys it.

1. NOTATION AND TERMINOLOGY

We consider only interactions governed by a differential equation $dx/dt = Ax$, where $x \in R^n$ and A is an $n \times n$ matrix. All matrices are real and are denoted A, B, A_0, A^-, etc. The *qualitative matrix* $Q(A)$ is the set of all matrices with the same sign pattern as A of $+$, $-$, and 0. That is, $Q(A) = \{B \in R^{n\times n}: \operatorname{sgn} b_{ij} = \operatorname{sgn} a_{ij} \text{ for } 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant n\}$.

$S(A)$, the *signed digraph* of A, has vertex set $V = \{1,2,\dots,n\}$ and edge set $E = E^+ \cup E^-$ with *positive edges* $E^+ = \{(i,j): a_{ij} > 0\}$ and *negative edges*

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$E^- = \{(i,j): a_{ij} < 0\}$. In drawing $S(A)$, positive edges are solid and negative edges are dashed. We will use the term *cycle* (k-cycle) to refer both to the sequence of k consecutive edges $(i_1, i_2)(i_2, i_3)\dots(i_k, i_{k+1})$ in $S(A)$ with $i_{k+1} = i_1$ but no other vertex repeated, and to the corresponding product of matrix entries $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$. We say the cycle is *positive* or *negative* depending on the sign of this product. For brevity cycles are denoted σ, σ_1, etc. If $\sigma = (i_1, i_2)\dots(i_{k-1}, i_k)(i_k, i_1)$, we define $V(\sigma) = \{i_1, i_2, \dots, i_k\}$.

Let the matrix A have characteristic polynomial $p_A(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n$, and let $H_2, \dots, H_{n-1}$ denote the Hurwitz determinants

$$\det H_k = \begin{bmatrix} c_1 & c_3 & c_5 & \cdot & \cdot & \cdot & 0 \\ 1 & c_2 & c_4 & & & & \cdot \\ 0 & c_1 & c_3 & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot \\ \cdot & & & & & \cdot & c_{k+1} \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & c_k \end{bmatrix}$$

with the convention $c_j = 0$ if $j > n$.

A is *stable* if all its eigenvalues have negative real part. A is *potentially stable* if some $B \in Q(A)$ is stable.

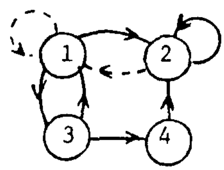
2. EXAMPLES

A. *A Matrix Which Is Potentially Stable If $a_{22} > 0$, Unstable If $a_{22} \leqslant 0$*

Let

$$Q(A) = \begin{bmatrix} - & + & + & 0 \\ - & + & 0 & 0 \\ + & 0 & 0 & + \\ 0 & + & 0 & 0 \end{bmatrix}.$$

The associated signed digraph $S(A)$ is



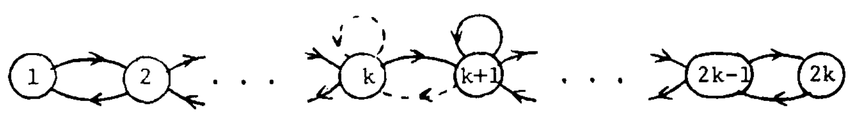
We apply the Routh-Hurwitz criterion, which states that A is stable if and only if the coefficients $c_1,\ldots,c_n$ and the Hurwitz determinants $H_2,\ldots,H_{n-1}$ are all positive [4]. Let $a, b, c, d, e > 0$ with $a_{11} = -a$, $a_{22} = b$, $a_{12}a_{21} = -c$, $a_{13}a_{31} = d$, and $a_{13}a_{34}a_{42}a_{21} = -e$. Then the characteristic polynomial coefficients are $c_1 = a - b$, $c_2 = -ab + c - d$, $c_3 = bd$, and $c_4 = e$, while the Hurwitz determinants are $H_2 = cc_1 + a(b^2 - ab - d)$ and $H_3 = c_3H_2 - c_1^2c_4$. These can all be made positive if $a > b$, c is large enough, and e is small enough. Thus A is potentially stable.

Now suppose we obtain the matrix B from A by changing the positive one-cycle $a_{22} > 0$ to $b_{22} = 0$ and leaving all other entries unchanged. Then for $Q(B)$ we will have $c_3 = 0$ and $H_3 = -c_1^2c_4 < 0$, so all matrices of this sign pattern will have an eigenvalue with nonnegative (in fact, positive) real part and are thus unstable. (Note that making $b_{22} < 0$ does not produce potential stability either.)

This provides a counterexample to one direction of Quirk's Proposition 2 in [6].

B. *Potentially Stable $n \times n$ Matrices of Each Order $n \geqslant 2$ with All but Two Cycles Nonnegative*

Let $k = \lfloor n/2 \rfloor$. That is, n is either $2k$ or $2k-1$. A is arbitrary except that we require $a_{kk} < 0$, $a_{k+1,k+1} > 0$, $a_{k,k+1}a_{k+1,k} < 0$, and $a_{i,i+1}a_{i+1,i} > 0$ for all $i \neq k$, $i = 1,\ldots,n-1$. For example, if n is even, A is tridiagonal, and all $a_{ii} = 0$ if $i \neq k, k+1$, then S(A) looks like



In Section 3 the matrix A is shown to be an instance of a construction which always produces potentially stable matrices.

C. *Matrices Which Are Potentially Stable When a 3-Cycle Is Positive, but Unstable When It Is Negative*

Let

$$Q(A) = \begin{bmatrix} - & + & 0 \\ 0 & 0 & + \\ + & - & 0 \end{bmatrix}.$$

Let $a_{11} = -a$, $a_{23}a_{32} = -b$, and $a_{12}a_{23}a_{31} = c$. The Routh-Hurwitz criterion shows A is stable when $ab > c$ (and all are positive). But if we reverse the sign of the 3-cycle so that $c < 0$, the resulting matrix is not stable. This is also generalized as a construction below.

3. SUFFICIENT CONDITIONS FOR POTENTIAL STABILITY

Given an $n \times n$ matrix A, we define a *skeleton* of A as any $n \times n$ matrix B for which $S(B)$ is a subgraph of $S(A)$. That is, we obtain $Q(B)$ from $Q(A)$ by setting an arbitrary number of entries equal to zero.

THEOREM 1. *If any skeleton of A is potentially stable, then A is potentially stable.*

Proof. The function $F: R^{n\times n} \to R^{2n-2}$ by $F(A) = (c_1, \dots, c_n, H_2, \dots, H_{n-1})$ is continuous, and therefore $F^{-1}(]0,\infty[^{2n-2})$ is an open set. By the Routh-Hurwitz criterion this is precisely the set of stable $n \times n$ matrices. If B is a skeleton of A, and $B_0 \in Q(B)$ is stable, then some neighborhood $N(B_0)$ is stable, and $N(B_0) \cap Q(A) \neq \phi$. Therefore, there exists stable $A_0 \in Q(A)$. ■

This is essentially the correct direction of Quirk's Proposition 2, the converse of which is shown false by the example in Section 2.A above.

We define a *compound k-cycle* to be a product of one or more vertex-disjoint cycles involving a total of k vertices. We say the compound cycle is of *even parity* if all but an even number of its factor cycles are negative; otherwise it is of *odd parity*. For a compound $\Sigma = \sigma_1 \cdots \sigma_r$ define $V(\Sigma) = V(\sigma_1) \cup \cdots \cup V(\sigma_r)$.

The following construction may be regarded as a procedure which starts with one potentially stable matrix (initially 1×1) and borders it (adds one row and one column) to produce another, repeating the process until a potentially stable matrix of the desired dimension is obtained.

CONSTRUCTION 1. We construct an $n \times n$ matrix A as follows:

(i) for $k = 1, 2, \dots, n$, A contains an even parity compound k-cycle Σ_k;
(ii) $V(\Sigma_1) \subset V(\Sigma_2) \subset \cdots \subset V(\Sigma_n)$;
(iii) for $k = 2, \dots, n$, whenever Σ_k contains a factor Σ' and $m < k$ is the smallest integer such that $\Sigma_1 \cdots \Sigma_m$ contains a (simple or compound cycle) factor Σ'' with $V(\Sigma') = V(\Sigma'')$, then $\Sigma' = \Sigma''$.

THEOREM 2. *The matrix A produced by Construction 1 is potentially stable.*

Proof. The proof depends upon two observations. First, the terms of any $k \times k$ principal minor of A are all compound k-cycles which appear with sign $(-1)^k$ if they are of even parity, $(-1)^{k+1}$ if of odd parity. Secondly, condition (iii) of Construction 1 allows us to make Σ_n the dominant term of $\det A$ by keeping the magnitude of each Σ_k sufficiently small relative to the magnitudes of its predecessors.

Assume the theorem is true whenever $n \geqslant N \geqslant 1$. Then for $n = N+1$, for convenience let $V(\Sigma_N) = \{1,\dots,N\}$ and partition

$$A = \begin{bmatrix} B & \vdots & * \\ \cdots & \cdots & \cdots \\ * & \vdots & a_{nn} \end{bmatrix}.$$

Since B is potentially stable by induction, let $B_0 \in Q(B)$ be stable and let $A_0(t) \in Q(A)$ be the corresponding matrix with B replaced by B_0 and with all the elements of the last column multiplied by the variable t. When $t=0$, $p_{A_0(t)}(\lambda) = \lambda p_{B_0}(\lambda)$, and $A_0(t)$ has eigenvalues $\lambda_1(A_0(t)) = \lambda_1(B_0),\dots,$ $\lambda_N(A_0(t)) = \lambda_N(B_0)$, all with negative real parts, and $\lambda_{N+1}(A_0(t)) = 0$. By continuity there exists $\varepsilon > 0$ such that when $0 < t < \varepsilon$, $\lambda_1(A_0(t)),\dots,$ $\lambda_N(A_0(t))$ are still stable and their product is still real with $\operatorname{sgn} \lambda_1(A_0(t)) \cdots \lambda_N(A_0(t)) = \operatorname{sgn} \det B_0 = (-1)^N$. Since $\det A_0(t)$ is arbitrarily close to Σ_{N+1}, which has sign $(-1)^{N+1}$, and since

$$\lambda_{N+1}(A_0(t)) = \frac{\det A_0(t)}{\lambda_1(A_0(t)) \cdots \lambda_N(A_0(t))},$$

this gives $\lambda_{N+1}(A_0(t)) < 0$. Thus $A_0(t) \in Q(A)$ is stable for all t (hence all Σ_{N+1}) sufficiently close to 0. ∎

This may be recognized as a variant of the Fisher-Fuller theorem [1–3].

We now use Construction 1 to show that the matrices in the example in Section 2.B above are potentially stable. Let $\Sigma_1 = a_{kk} < 0$ and $\Sigma_2 = a_{k,k+1}a_{k+1,k} < 0$. The resulting 2×2 submatrix is potentially stable by Theorem 2, and by Theorem 1 it is still potentially stable when we add the positive 1-cycle $a_{k+1,k+1} > 0$. Since $V(\Sigma_2) = V(\Sigma_1 a_{k+1,k+1})$, condition (iii) of the construction requires that Σ_2 be used in all subsequent cycles with a factor in vertices k and $k+1$. Let $\Sigma_3 = (a_{k-1,k}a_{k,k-1})(a_{k+1,k+1})$ and $\Sigma_4 =$

$(a_{k-1,k}a_{k,k-1})(a_{k+1,k+2}a_{k+2,k+1})$. Each is a product of two positive cycles and therefore of even parity. For $r=5,6,\ldots,n$, let

$$V(\Sigma_r)=\left\{k-\left\lfloor\frac{r-1}{2}\right\rfloor,\ldots,k+\left\lfloor\frac{r}{2}\right\rfloor\right\}.$$

Notice that the extreme vertices of $V(\Sigma_r)$ together with condition (iii) unambiguously determine Σ_r as the product of the two extreme positive 2-cycles and Σ_{r-4}. Thus, by the construction the resulting matrix is potentially stable.

Construction 2. Given an $(n-2)\times(n-2)$ matrix A^-, we construct an $n\times n$ matrix A as follows:

(i) let A^- appear as a principal submatrix of A;
(ii) the 2×2 principal submatrix complementary to A^- contains a negative 2-cycle;
(iii) A contains a positive n-cycle.

Theorem 3. *If A^- is potentially stable, the matrix A produced by Construction 2 is also potentially stable.*

Proof. For convenience suppose $A^-=A_0^-\in Q(A^-)$ is itself stable. Suppose the positive n-cycle is a multiple of $t>0$, and first set $t=0$. When the n-cycle is not present, the eigenvalues of A are just the eigenvalues of A^- together with those of the complementary 2×2 submatrix. By continuity there exists $\varepsilon>0$ such that when $0<t<\varepsilon$ we may continuously factor

$$p_A(\lambda)=\left(\lambda^{n-2}+a_1\lambda^{n-3}+\cdots+a_{n-2}\right)\left(\lambda^2+b_1\lambda+b_2\right),$$

so that the roots of the first factor remain stable. When $t=0$, the first factor is $p_{A^-}(\lambda)$, $b_1=0$, and b_2 is the absolute value of the negative 2-cycle. As t increases from 0 the conjugate imaginary roots of the second factor become stable if b_1 becomes positive. At the same time the growing positive n-cycle decreases c_n. Thus, it suffices to show

$$\left.\frac{\partial b_1}{\partial c_n}\right|_{t=0}<0.$$

We readily compute

$$\left.\frac{\partial(c_1,\ldots,c_n)}{\partial(a_1,\ldots,a_{n-2};b_1,b_2)}\right|_{t=0}$$

$$=\begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 \\ 0 & 1 & \cdot & & & & a_1 & 1 \\ b_2 & 0 & \cdot & \cdot & & & a_2 & a_1 \\ 0 & b_2 & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & & & & \cdot & 0 & a_{n-2} & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 & b_2 & 0 & a_{n-2} \end{bmatrix}.$$

$\partial b_1/\partial c_n|_{t=0}$ is the $(n-1,n)$ entry of the inverse matrix. Assuming the magnitude of the negative 2-cycle is sufficiently small, we can ignore b_2 and obtain

$$\operatorname{sgn}\left.\frac{\partial b_1}{\partial c_n}\right|_{t=0}=\operatorname{sgn}\frac{-a_{n-3}}{a_{n-2}^2}=-1. \qquad \blacksquare$$

When $n=3$, this gives us the example in Section 2.C above.

We may modify Construction 1 to use it in conjunction with Construction 2. But we must then narrowly restrict the kind of even parity compound cycles we allow. This leads to

CONSTRUCTION 1′. Given an $(n-1)\times(n-1)$ matrix A^-, we construct an $n\times n$ matrix A as follows:

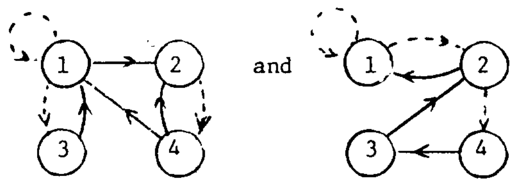
(i) let A^- appear as a principal submatrix of A;
(ii) A contains a negative n-cycle.

A proof similar to that of Theorem 2 shows that the resulting matrix A is potentially stable if A^- is potentially stable.

4. SOME OPEN QUESTIONS

In view of the counterexamples given here, what is a necessary and sufficient condition for a stable matrix to remain stable upon the addition of a negative diagonal matrix? One merely sufficient condition is that there exists positive diagonal D such that $DA + A^T D$ is negative definite.

It is known that even for 4×4 matrices the above constructions do not account for skeletons of all potentially stable matrices. For example, both



represent potentially stable matrices. What are some other constructions? In particular, how should one join a potentially stable submatrix to a "neutral" complementary submatrix with pure imaginary eigenvalues so that the resulting matrix is potentially stable?

Under what conditions does Construction 1 alone account for all potentially stable matrices? More generally, for what class of matrices does A potentially stable imply some $(n-1)\times(n-1)$ principal submatrix of A is potentially stable?

What is the computational complexity of recognizing potentially stable matrices? Tarski's decision procedure for elementary algebra [8] together with the Routh-Hurwitz criterion provides a finite algorithm, but it is not known how efficient such an algorithm can be. A related problem is to find some positive function f such that if A is an $n \times n$ potentially stable matrix, then there exists stable $B \in Q(A)$ all of whose entries are integers between $-f(n)$ and $f(n)$. If one could show $\log f$ to be polynomial, it would follow that the complexity of recognizing potentially stable matrices is no worse than nondeterministic polynomial time.

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